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Note on Mr. A. Y. G. Campbell's paper "On the Variation of Uncanonical Arbitrary Constants, with an Application to the Planetary Theory." By Sir Robert S. Ball, LL.D., F.R.S.

The object of this note is to set down another method by which the calculation of Lagrange's "parentheses" in the planetary theory can be effected very simply. The method was suggested by a manuscript which Mr. Campbell kindly communicated to me. I have also to acknowledge the assistance derived from the instructive remarks of Mr. Cowell and Mr. Whittaker, to whom, as well as to Mr. Campbell, it has been my privilege to have lectured at Cambridge.

Adopting the notation of Mr. Campbell's manuscript, we put

$$\frac{\partial(x,\dot{x})}{\partial(\lambda,\mu)} \text{ for } \frac{dx}{d\lambda}\cdot\frac{d\dot{x}}{d\mu}-\frac{dx}{d\mu}\cdot\frac{d\dot{x}}{d\lambda},$$

where λ, μ are any two of the six arbitrary constants.

We thus have the expression for the "parenthesis"

$$[\lambda,\mu]=\frac{\partial(x,\dot{x})}{\partial(\lambda,\mu)}+\frac{\partial(y,\dot{y})}{\partial(\lambda,\mu)}+\frac{\partial(z,\dot{z})}{\partial(\lambda,\mu)}.$$

We now make the transformation to another set of rectangular coordinates x_1, y_1, z_1, where

$$\begin{aligned} x &= l_1x_1 + l_2y_1 + l_3z_1 \\ y &= m_1x_1 + m_2y_1 + m_3z_1 \\ z &= n_1x_1 + n_2y_1 + n_3z_1 \end{aligned}$$

It is unnecessary to write down the general result. For the planetary theory we take the special case where

$$\begin{array}{lll} l = \cos\alpha & l_2 = -\sin\alpha\cos\phi & l_3 = 0 \\ m_1 = \sin\alpha & m_2 = \cos\alpha\cos\phi & m_3 = 0 \\ n_1 = 0 & n_2 = \sin\phi & n_3 = 0 \end{array}$$

And if we express x_1 by ξ and y_1 by η we get after a little reduction

$$[\lambda,\mu]=\frac{\partial(\xi,\dot{\xi})}{\partial(\lambda,\mu)}+\frac{\partial(\eta,\dot{\eta})}{\partial(\lambda,\mu)}-\sin\phi\,(\xi\dot{\eta}-\dot{\xi}\eta)\frac{\partial(\alpha,\phi)}{\partial(\lambda,\mu)}$$

$$+\cos\phi\frac{d\alpha}{d\lambda}\frac{d}{d\mu}(\xi\dot{\eta}-\dot{\xi}\eta)-\cos\phi\frac{d\alpha}{d\mu}\frac{d}{d\lambda}(\xi\dot{\eta}-\dot{\xi}\eta)$$

We now transform to two new variables X and Y given by the equations

$$\begin{aligned} \xi &= \mathrm{X}\cos g-\mathrm{Y}\sin g \\ \eta &= \mathrm{X}\sin g+\mathrm{Y}\cos g \end{aligned}$$

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Also from the known properties of elliptic motion

$$\xi\dot{\eta}-\eta\dot{\xi}=\sqrt{a(1-e^2)}=h,$$

whence the general symbolic representation of a "parenthesis"

$$[\lambda, \mu]=\frac{\partial(\mathrm{X}, \dot{\mathrm{X}})}{\partial(\lambda, \mu)}+\frac{\partial(\mathrm{Y}, \dot{\mathrm{Y}})}{\partial\lambda, \mu)}-h \sin \phi \frac{\partial(\alpha, \phi)}{\partial(\lambda, \mu)}+\cos \phi \frac{\partial(\alpha, h)}{\partial(\lambda, \mu)}+\frac{\partial(g, h)}{\partial(\lambda, \mu)}.$$

It will now be convenient to divide the fifteen "parentheses" into two groups, those which contain ϵ and those which do not. Taking the former first, we have

$$[\epsilon, \mu]=\frac{\partial(\mathrm{X}, \dot{\mathrm{X}})}{\partial(\epsilon, \mu)}+\frac{\partial(\mathrm{Y}, \dot{\mathrm{Y}})}{\partial(\epsilon, \mu)},$$

for obviously the three latter terms in the general expression do not contribute.

As ϵ only occurs in the expression $a^{-\frac{3}{2}}t+\epsilon$, and as t is never found except in this expression

$$\frac{d\mathrm{X}}{d\epsilon}=a^{\frac{3}{2}}\dot{\mathrm{X}} \qquad \frac{d\mathrm{Y}}{d\epsilon}=a^{\frac{3}{2}}\dot{\mathrm{Y}}$$

$$\frac{d\dot{\mathrm{X}}}{d\epsilon}=a^{\frac{3}{2}}\ddot{\mathrm{X}} \qquad \frac{d\dot{\mathrm{Y}}}{d\epsilon}=a^{\frac{3}{2}}\ddot{\mathrm{Y}}$$

whence we easily find

$$\frac{d\dot{\mathrm{X}}}{d\epsilon}=-\frac{a^{\frac{3}{2}}}{\rho^3}\mathrm{X}\,; \ \frac{d\dot{\mathrm{Y}}}{d\epsilon}=-\frac{a^{\frac{3}{2}}}{\rho^3}\mathrm{Y}$$

We thus have

$$[\epsilon, \mu]=\frac{1}{2}\,a^{\frac{3}{2}}\,\frac{d}{d\mu}(\dot{\mathrm{X}}^2+\dot{\mathrm{Y}}^2)+\frac{1}{2}\,a^{\frac{3}{2}}\,\frac{1}{\rho^3}\frac{d}{d\mu}(\mathrm{X}^2+\mathrm{Y}^2)=\frac{1}{2}\,a^{-\frac{1}{2}}\frac{da}{d\mu}.$$

For those parentheses $[\lambda, \mu]$ whereof neither is ϵ the following considerations will simplify the result.

In the values of X, $\dot{\mathrm{X}}$, Y, $\dot{\mathrm{Y}}$, t and ϵ enter only by powers of the exponential $e^{\,i\,(a^{-\frac{3}{2}}t+\epsilon)}$, where e is the base of natural logs. If this be differentiated with respect to constants which are neither a nor ϵ, this exponential is unaffected. But we know that the time disappears from the parenthesis; therefore it is immaterial what be the value of $a^{-\frac{3}{2}}t+\epsilon$, and we may accordingly take both t and ϵ zero. If, however, the exponential be differentiated with respect to a, then we have terms like $te^{i\,(a^{-\frac{3}{2}}t+\epsilon)}$. But such terms must be mutually destructive, for the exponential terms simply can never annul them. We need therefore never have formed such terms. Hence we may *before* differentiation make both t and ϵ zero. Hence both $\dot{\mathrm{X}}$ and $\dot{\mathrm{Y}}$ may be treated as non-existent, and

$$\frac{\partial(\mathrm{X}, \dot{\mathrm{X}})}{\partial(\lambda, \mu)}+\frac{\partial(\mathrm{Y}, \dot{\mathrm{Y}})}{\partial(\lambda, \mu)}$$

will be zero whenever ϵ is neither λ nor μ.

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We have thus reduced the formulæ for the parentheses to the simple forms following :—

$$[\epsilon, \mu] = \frac{1}{2} a^{-\frac{1}{2}} \frac{da}{d\mu} \text{ for all which contain } \epsilon$$

$$[g, \mu] = \frac{\partial(g, h)}{\partial(g, \mu)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad g$$

$$[\phi, \mu] = h \sin \phi \frac{\partial(\phi, \alpha)}{\partial(\phi, \mu)} \quad . \quad . \quad . \quad . \quad . \quad \phi$$

$$[\lambda, \mu] = \cos \phi \frac{\partial(\alpha, h)}{\partial(\lambda, \mu)} \text{ for all which contain only } a, e, \alpha$$

Perhaps the most instructive form is that which comprises *all* the fifteen parentheses in the general expression

$$[\lambda, \mu] = \tfrac{1}{2} a^{-\frac{1}{2}} \frac{\partial(\epsilon, a)}{\partial(\lambda, \mu)} + \frac{\partial(g, h)}{\partial(\lambda, \mu)} + \cos \phi \frac{\partial(\alpha, h)}{\partial(\lambda, \mu)} - h \sin \phi \frac{\partial(\alpha, \phi)}{\partial(\lambda, \mu)}.$$

We thus obtain by inspection

$$[\epsilon, a] = \frac{1}{2} a^{-\frac{1}{2}} \qquad [\epsilon, e] = 0 \qquad [\epsilon, \alpha] = 0 \qquad [\epsilon, \phi] = 0 \qquad [\epsilon, g] = 0$$

$$[g, a] = \frac{dh}{da} \qquad [g, e] = \frac{dh}{de} \qquad [g, \alpha] = 0 \qquad [g, \phi] = 0$$

$$[\phi, a] = 0 \qquad [\phi, e] = 0 \qquad [\phi, \alpha] = h \sin \phi$$

$$[e, a] = 0 \qquad [\alpha, a] = \cos \phi \frac{dh}{da} \qquad [\alpha, e] = \cos \phi \frac{dh}{de}$$

where

$$h = \sqrt{a(1 - e^2)}; \quad \frac{dh}{da} = \frac{1}{2} a^{-\frac{1}{2}} \sqrt{1 - e^2}; \quad \frac{dh}{de} = -\frac{a^{\frac{1}{2}} e}{\sqrt{1 - e^2}}$$

thus giving the well-known expressions in the Planetary Theory.

On Differential Refraction to Terms of Higher Orders than the First. By H. H. Turner, M.A., B.Sc., Savilian Professor.

1. For moderate zenith distances the differential refraction on, say, a photographic plate is expressed with sufficient accuracy by the first powers of the relative coordinates when the field is not too large ; but a plate may be exposed at a considerable zenith distance for the purpose of measuring parallax or some other reason, and again the field photographed may be of considerable extent. In such cases the first powers of the coordinates no longer afford a sufficiently accurate approximation, and we must take in